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Polynomials, fragments of temporal logic and the variety **DA** over traces

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Abstract

We show that some language-theoretic and logical characterizations of recognizable word languages whose syntactic monoid is in the variety **DA** also hold over traces. To this end we give algebraic characterizations for the language operations of generating the polynomial closure and generating the unambiguous polynomial closure over traces.

We also show that there exist natural fragments of local temporal logic that describe this class of languages corresponding to **DA**. All characterizations are known to hold for words.

Keywords: Mazurkiewicz traces; Polynomial closure; Temporal logic

1. Introduction

The concept of partially commutative free monoids was introduced by Keller as a generalization of free monoids in order to describe the behavior of parallel programs [4]. Later, Mazurkiewicz established the notion of *trace* monoids for these structures [5,6]. Since then traces, the elements of trace monoids, have become a rather popular setting to study concurrency. Many aspects of traces and trace languages have been researched; see *The Book of Traces* [1] for an overview.

Over words it has turned out that finite semigroups are a powerful technique to refine the class of recognizable languages [2]. Two natural operations on classes of languages are the polynomial closure and the unambiguous polynomial closure. For particular classes of languages, so-called language varieties, it has been shown that these operations admit algebraic counterparts in terms of the so-called Mal'cev product [12]. In Section 3 (resp. Section 4) we will show that this correspondence between the Mal'cev product and the polynomial closure (resp. the unambiguous polynomial closure) for restricted varieties also holds over traces.

In Section 5 we tighten these results in the particular case of the class **DA** of finite monoids to get two language-theoretic characterizations of the class of trace languages whose syntactic monoid is in **DA**. In Section 6 we show that over traces the fragments of local temporal logic $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}]$, $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{M}]$ and $\mathrm{TL}[\mathsf{X}_a, \mathsf{Y}_a]$ also express exactly these languages. All three characterizations are known to hold for words [13,14].

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2. Preliminaries

A set S is a *semigroup* if it is equipped with an associative binary operation. The set S forms a *monoid* if it is a semigroup and if there exists a neutral element, i.e., an element denoted by 1 and satisfying $1a = a = a1$ for all $a \in S$. An element e of a semigroup is called *idempotent* if $e^2 = e$. A mapping $\eta : S \to T$ between two semigroups S and T is a *semigroup homomorphism* if $\eta(ab) = \eta(a)\eta(b)$ for all $a, b \in S$. If furthermore S and T are monoids and $\eta(1) = 1$, then η is *monoid homomorphism*. A relation $\tau \subseteq S \times T$ is a *relational semigroup morphism* between semigroups S and T if $\tau(a) \neq \emptyset$ and $\tau(ab) \subseteq \tau(a)\tau(b)$ for all $a, b \in S$ where $\tau(a) = \{\, c \in T \mid (a, c) \in \tau \,\}$. In the context of monoids we additionally require $1 \in \tau(1)$ and then τ is called a *relational monoid morphism*. If there is no confusion or if the statement holds in either case we omit the terms *relational*, *semigroup* and *monoid* and only use the words *morphism* and *homomorphism*. As for functional homomorphisms, we also use the notation $\tau : S \to T$ for morphisms. For two (homo)morphisms $\eta : S \to T$ and $\nu : S \to R$ we define their *product* $\eta \times \nu : S \to T \times R : a \mapsto (\eta(a), \nu(a))$.

The *graph* of a morphism $\tau : S \to T$ is defined as $\mathrm{graph}(\tau) = \{\, (a, c) \mid c \in \tau(a) \,\}$. It is easy to see that $\mathrm{graph}(\tau)$ is a subsemigroup (resp. submonoid) of $S \times T$. For any relational morphism $\tau : S \to T$ the projections $\pi_1 : \mathrm{graph}(\tau) \to S$ and $\pi_2 : \mathrm{graph}(\tau) \to T$ satisfy $\tau(a) = \pi_2(\pi_1^{-1}(a))$ for all $a \in S$, i.e., $\tau = \pi_2 \circ \pi_1^{-1}$. The condition $\tau(a) \neq \emptyset$ for all $a \in S$ implies that π_1 is onto. In fact, whenever we have two homomorphisms $\alpha : R \to S$ and $\beta : R \to T$ and α is onto, the composition $\beta \circ \alpha^{-1} : S \to T$ forms a relational morphism [8].

An *ordered semigroup* is a semigroup S equipped with a partial order relation $\leq$ such that $a \leq b$ implies $ca \leq cb$ and $ac \leq bc$ for all $a, b, c \in S$. Every semigroup S forms also an ordered semigroup $(S, =)$. For homomorphisms between ordered semigroups $\eta : (S, \leq) \to (T, \preceq)$ we additionally require that $a \leq b$ implies $\eta(a) \preceq \eta(b)$ for all $a, b \in S$. More details can be found in [9].

We are interested in the interplay between classes of finite monoids and classes of recognizable subsets of infinite monoids. The connection between them is the syntactic congruence. Let L be a subset of a monoid $\mathbb{M}$. Then the *syntactic congruence* $\sim_L \subseteq \mathbb{M} \times \mathbb{M}$ of L is defined by

$$p \sim_L q \Leftrightarrow (\forall u, v \in \mathbb{M}\colon upv \in L \Leftrightarrow uqv \in L)\,.$$

The natural homomorphism $\mu_L : \mathbb{M} \to \mathbb{M}/{\sim_L} : p \mapsto [p]_{\sim_L}$ is called the *syntactic homomorphism* of L. The monoid $M(L) = \mathbb{M}/{\sim_L}$ is called the *syntactic monoid* of L. The *syntactic quasiordering* $\leq_L$ of L is defined by $p \leq_L q \Leftrightarrow (\forall u, v \in \mathbb{M}\colon uqv \in L \Rightarrow upv \in L)$. The relation $\leq_L$ induces a partial order on $M(L)$ such that $(M(L), \leq_L)$ forms an ordered monoid. It is called the *syntactic ordered monoid* of L.

Equations are a tool to describe classes of finite semigroups. Let Ω be a finite set and let $w, v \in \Omega^+$ (resp. Ω^* for monoids). A semigroup S *satisfies* the equation $w = v$, if for all homomorphisms $\eta : \Omega^+ \to S$ we have $\eta(w) = \eta(v)$. In a finite semigroup the unique idempotent power of an element a is denoted by a^ω. We also allow the ω-operator in equalities and define $\eta(w^\omega) = \eta(w)^\omega$. By $[\![\, w = v \,]\!]$ we denote the class of finite semigroups (resp. finite monoids) satisfying $w = v$. Analogously, we can define the class of finite ordered semigroups satisfying an inequality $w \leq v$.

The next tool we will need in order to define classes of finite semigroups is the Mal'cev product. Let $\mathbf{V}$ and $\mathbf{W}$ be two classes of finite semigroups. A semigroup S is contained in the *Mal'cev product* $\mathbf{W} \circledM \mathbf{V}$ of $\mathbf{V}$ by $\mathbf{W}$ if there exists a semigroup $T \in \mathbf{V}$ and a relational morphism $\tau : S \to T$ such that for each idempotent $e \in T$ the set $\tau^{-1}(e)$ forms a semigroup in $\mathbf{W}$.

Let Σ be a finite alphabet and $I \subseteq \Sigma \times \Sigma$ be a symmetric and irreflexive relation. The *trace monoid* generated by (Σ, I) is the quotient $\mathbb{M}(\Sigma, I) = \Sigma^*/\sim_I$ where $\sim_I$ is the congruence generated by $\{\, (ab, ba) \mid (a, b) \in I \,\}$. The elements of $\mathbb{M}(\Sigma, I)$ are called *traces*, I is called the *independence relation* and $D = \Sigma^2 \setminus I$ is the *dependence relation*. Let $w \in \Sigma^*$. By $[w]_I$ we denote the trace $[w]_I = \{\, v \in \Sigma^* \mid v \sim_I w \,\}$. The word $w \in \Sigma^*$ is called a *word representative* of a trace $t \in \mathbb{M}(\Sigma, I)$ if $t = [w]_I$. As for words, $|t| \in \mathbb{N}$ is the *length* of the trace $t \in \mathbb{M}(\Sigma, I)$ and $\mathrm{alph}(t) \subseteq \Sigma$ is its *alphabet*.

Let w be a word representative of a trace t. With t we can associate a graph $(V_t, <_t, \mathrm{label}_t)$ where $V_t = \{\, \nu \mid \nu \text{ is a position of } w \,\}$ is the set of vertices and

$$\mathrm{label}_t : V_t \to \Sigma : \nu \mapsto \text{“letter of } w \text{ at position } \nu\text{”}$$

is a labeling of the vertices. Let

$$E_t = \{\, (\nu, \chi) \in V_t^2 \mid \nu \text{ occurs before } \chi \text{ in } w \text{ and } (\mathrm{label}_t(\nu), \mathrm{label}_t(\chi)) \in D \,\}.$$

The set of edges $<_t$ is now defined as the transitive closure of E_t. The relation $<_t$ is a (strict) partial order on V_t. Up to isomorphism, the definition of this graph is independent of the choice of the word representative. A *linearization* of $<_t$ is a (strict) total order on V_t that contains $<_t$. The linearizations of $<_t$ yield exactly the word representatives of t. Therefore, by abuse of notation we will identify the word representative w, the trace t and its graph $(V_t, <_t, \text{label}_t)$.

3. Polynomial closure

In the following, we fix the trace monoid $\mathbb{M}(\Sigma, I)$ over a non-empty finite alphabet Σ. A class of finite monoids $\mathbf{V}$ is called a *variety* if it is closed under taking finite products, submonoids and homomorphic images [8]. We will also use this notion if $\mathbf{V}$ is a class of finite ordered monoids [9]. By **Com** we denote the class of finite commutative monoids $[\![\, xy = yx \,]\!]$ and by $\mathbf{J_1}$ we denote the class of idempotent and commutative monoids $[\![\, x^2 = x \,]\!] \cap \mathbf{Com}$.

Lemma 3.1. *Let* $\mathbf{V}$ *be a variety of monoids with* $\mathbf{J_1} \subseteq \mathbf{V}$ *and let* $M_0, \ldots, M_n \in \mathbf{V}$. *For all* $i \in \{0, \ldots, n\}$ *let* $\mu_i : \mathbb{M}(\Sigma, I) \to M_i$ *be homomorphisms. Then there exists a monoid* $N \in \mathbf{V}$ *and a homomorphism* $\eta : \mathbb{M}(\Sigma, I) \to N$ *such that for all* $x, y \in \mathbb{M}(\Sigma, I)$ *satisfying* $\eta(x) = \eta(y)$ *the following conditions hold:*

(a) *For all homomorphisms* μ_i, $0 \le i \le n$, *we have* $\mu_i(x) = \mu_i(y)$.
(b) $\text{alph}(x) = \text{alph}(y)$.
(c) *Let* x' *and* y' *be connected components of* x *and of* y *respectively such that* $\text{alph}(x') = \text{alph}(y')$. *Then we have* $\eta(x') = \eta(y')$.
(d) *If* $\eta(x)$ *is idempotent then* $\eta(x')$ *is idempotent for every connected component* x' *of* x.

Proof. The power set 2^Σ of Σ forms a commutative and idempotent monoid under the union operation. We set

$$N = 2^\Sigma \times \prod_{\Gamma \subseteq \Sigma} M_0 \times \cdots \times M_n.$$

Since $2^\Sigma \in \mathbf{J_1} \subseteq \mathbf{V}$ and since $\mathbf{V}$ is a variety, we have $N \in \mathbf{V}$. Next we define

$$\eta = \text{alph} \times \prod_{\Gamma \subseteq \Sigma} ((\mu_0 \times \cdots \times \mu_n) \circ \pi_\Gamma) : \mathbb{M}(\Sigma, I) \to N,$$

where π_Γ is the natural projection $\mathbb{M}(\Sigma, I) \to \mathbb{M}(\Gamma, \Gamma^2 \cap I) : x \mapsto \pi_\Gamma(x)$. Note that $\mathbb{M}(\Gamma, \Gamma^2 \cap I) \subseteq \mathbb{M}(\Sigma, I)$. Condition (a) is verified in the components of N and ν corresponding to $\Gamma = \Sigma$ and condition (b) is fulfilled by reason of the first component. Let x' and y' be connected components of x and of y with $\text{alph}(x') = \text{alph}(y')$. Let $\Gamma \subseteq \Sigma$ and let $i \in \{0, \ldots, n\}$. To conclude (c) we have to show $\mu_i(\pi_\Gamma(x')) = \mu_i(\pi_\Gamma(y'))$. Since $x' = \pi_{\text{alph}(x')}(x)$ we have $\pi_\Gamma(x') = \pi_\Gamma(\pi_{\text{alph}(x')}(x)) = \pi_{\Gamma'}(x)$ with $\Gamma' = \Gamma \cap \text{alph}(x')$. A similar argument for y' and y and $\eta(x) = \eta(y)$ implies $\mu_i(\pi_\Gamma(x')) = \mu_i(\pi_{\Gamma'}(x)) = \mu_i(\pi_{\Gamma'}(y)) = \mu_i(\pi_\Gamma(y'))$. Now let $\eta(x)$ be idempotent. This means that every component of $\eta(x)$ is idempotent and since every component of $\eta(x')$ is also a component of $\eta(x)$, we have that $\eta(x')$ is also idempotent. $\square$

We say that a trace t_1 is a *factor* of a trace t_2 if there exist traces s_1 and s_2 such that $t_2 = s_1 t_1 s_2$.

Lemma 3.2. *Let* $a \in \Sigma$, *let* $t_0, t_1 \in \mathbb{M}(\Sigma, I)$ *and let* $x \in \mathbb{M}(\Sigma, I)$ *be connected. If* $x^{|\Sigma|+m}$ *is a factor of* $t_0 a t_1$ *for* $m \in \mathbb{N}$ *then there exist* $m_0, m_1 \in \mathbb{N}$ *such that* $m_0 + m_1 = m$ *and* x^{m_i} *is a factor of* t_i *for* $i = 0$ *and* $i = 1$.

Proof. The proof is similar to the proof that x^* is recognizable if $x \in \mathbb{M}(\Sigma, I)$ is connected [7, Proposition 6.3.11]. Since x is connected, between any two letters of $\text{alph}(x)$ we have an undirected path in the dependence graph (Σ, D) of length at most $|\Sigma|$ such that all vertices on this path are in $\text{alph}(x)$. Directed paths following the same labels also exist in $x^{|\Sigma|}$ between all vertices of the first x and all vertices of the last x in this product. There could be some x's that have vertices in t_0 as well as vertices in $a t_1$. The above argument shows that starting with the first x with this property we could lose at most $|\Sigma| - 1$ many of the x's of $x^{|\Sigma|+m}$ as factors of t_0 or $a t_1$. The letter a could be a factor of one x. It follows that there remain m many x's as factors of either t_0 or t_1. $\square$

For a class $\mathcal{V}$ of trace languages over $\mathbb{M}(\Sigma, I)$ we define the *polynomials* $\text{Pol}\,\mathcal{V}$ over $\mathcal{V}$ as the trace languages that are finite unions of languages of the form

$$L_0 a_1 L_1 \cdots a_n L_n$$

where $n \in \mathbb{N}$ and $L_i \in \mathcal{V}$ for all $0 \leq i \leq n$. We say that a class $\mathcal{V}$ of trace languages over $\mathbb{M}(\Sigma, I)$ *corresponds* to a class of monoids $\mathbf{V}$, if $\mathcal{V} = \{ L \subseteq \mathbb{M}(\Sigma, I) \mid M(L) \in \mathbf{V} \}$ or $\mathcal{V} = \{ L \subseteq \mathbb{M}(\Sigma, I) \mid (M(L), \leq_L) \in \mathbf{V} \}$ for ordered monoids $\mathbf{V}$, respectively.

Theorem 3.3. *Let $\mathbf{V}$ be a variety of monoids such that $\mathbf{J_1} \subseteq \mathbf{V}$ and let $\mathcal{V}$ be the class of trace languages corresponding to $\mathbf{V}$. Then the syntactic ordered monoid of every language $L \in \operatorname{Pol} \mathcal{V}$ is in $[\![x^\omega y x^\omega \leq x^\omega]\!] ⓜ \mathbf{V}$.*

Proof. We modify the proof for words in [12]. Let $L = L_0 a_1 L_1 \cdots a_n L_n$, where $L_i \in \mathcal{V}$ for all $0 \leq i \leq n$. Let $\eta : \mathbb{M}(\Sigma, I) \to N$ be as in Lemma 3.1 with $M_i = M(L_i)$ for $i \in \{0, \ldots, n\}$. Let $(M(L), \leq_L)$ be the syntactic ordered monoid of L and $\mu : \mathbb{M}(\Sigma, I) \to M(L)$ its syntactic homomorphism. We obtain the relational morphism $\tau = \eta \circ \mu^{-1} : M(L) \to N$.

Let $e \in N$ be idempotent, let $x, y, u, v \in \mathbb{M}(\Sigma, I)$ such that $\eta(x) = e = \eta(y)$ and let $m \geq n\,|\Sigma| + 1$. The trace x can be decomposed into connected components $x = x_1 \cdots x_\ell$ such that $\operatorname{alph}(x_i) \times \operatorname{alph}(x_j) \subseteq I$ for all $1 \leq i \neq j \leq \ell$. Lemma 3.1(b) implies $\operatorname{alph}(x) = \operatorname{alph}(y)$. Hence, the trace y can also be decomposed into connected components $y = y_1 \cdots y_\ell$ such that $\operatorname{alph}(y_j) = \operatorname{alph}(x_j)$ for all $1 \leq j \leq \ell$. Suppose $ux^m v \in L$. By applying Lemma 3.2 up to n times we can conclude that for every $j \in \{1, \ldots, \ell\}$ there exists $i \in \{0, \ldots, n\}$ and a factorization $ux^m v = z_0 z_1 x_j z_2 z_3$ such that

$$\begin{aligned} z_0 &\in L_0 a_1 L_1 \cdots L_{i-1} a_i \\ z_1 x_j z_2 &\in L_i \\ z_3 &\in a_{i+1} L_{i+1} \cdots a_n L_n. \end{aligned}$$

By Lemma 3.1 we have that $\mu_i(x_j) = \mu_i(y_j)$ is idempotent and therefore we have $z_1 x_j x_j^{k_1} y_j x_j^{k_2} z_2 \in L_i$ for all $k_1, k_2 \in \mathbb{N}$. By applying this pumping argument to all connected components of x, by a suitable choice of the exponents we can conclude $ux^m y x^m v \in L$.

Thus for all $m \geq n\,|\Sigma| + 1$ we have $\mu(x^m y x^m) \leq_L \mu(x^m)$ and therefore $\tau^{-1}(e) \in [\![x^\omega y x^\omega \leq x^\omega]\!]$. This shows $(M(L), \leq_L) \in [\![x^\omega y x^\omega \leq x^\omega]\!] ⓜ \mathbf{V}$. This Mal'cev product forms a variety of ordered semigroups. Language classes corresponding to varieties are closed under finite unions. Hence, we can conclude that $\operatorname{Pol} \mathcal{V}$ is a subset of the trace languages corresponding to $[\![x^\omega y x^\omega \leq x^\omega]\!] ⓜ \mathbf{V}$. $\square$

By $\pi_I : \Sigma \to \mathbb{M}(\Sigma, I) : w \mapsto [w]_I$ we denote the canonical projection.

Lemma 3.4. *Let $\eta : \Sigma^* \to M$ be a homomorphism from Σ^* to a commutative monoid M. Then there exists a unique homomorphism $\nu : \mathbb{M}(\Sigma, I) \to M$ such that $\nu \circ \pi_I = \eta$.*

Proof. We define $\nu(a) = \eta(a)$ for all $a \in \Sigma$. Since M is commutative, ν can be uniquely extended to a homomorphism $\mathbb{M}(\Sigma, I) \to M$. By definition of ν we have $\nu([w]_I) = \eta(w)$ for all $w \in \Sigma^*$. Since $\zeta \circ \pi_I = \eta$ implies $\zeta(a) = \eta(a)$ for all $a \in \Sigma$, we have that ν is unique. $\square$

The proof of the following lemma can be found in [2] as a special case of Proposition 1.1, page 186.

Lemma 3.5. *The syntactic monoids of a trace language $L \subseteq \mathbb{M}(\Sigma, I)$ and its corresponding word language $\pi_I^{-1}(L) \subseteq \Sigma^*$ are isomorphic.*

Let $\mathbf{V}$ be a variety of finite monoids. We say that $\mathcal{V}$ is the corresponding $*$-*variety* if $\mathcal{V} = \{ K \subseteq \Sigma^* \mid M(K) \in \mathbf{V} \}$. As for classes of trace languages, we define $\operatorname{Pol} \mathcal{V}$ as the (word) languages that are finite unions of languages of the form $K_0 a_1 K_1 \cdots a_n K_n$, where $n \in \mathbb{N}$ and $K_i \in \mathcal{V}$ for all $0 \leq i \leq n$. For a partial converse of Theorem 3.3 we will use the following theorem from [12].

Theorem 3.6 (*Pin and Weil [12]*). *Let $\mathbf{V}$ be a variety of finite monoids, let $\mathcal{V}$ be the corresponding $*$-variety and let $K \subseteq \Sigma^*$. If the syntactic ordered monoid of K is in $[\![x^\omega y x^\omega \leq x^\omega]\!] ⓜ \mathbf{V}$, then $K \in \operatorname{Pol} \mathcal{V}$.*

For commutative varieties we can state this theorem for traces.

Theorem 3.7. *Let $\mathbf{V} \subseteq \mathbf{Com}$ be a variety of finite commutative monoids, let $\mathcal{V}$ be the class of trace languages corresponding to $\mathbf{V}$ and let $L \subseteq \mathbb{M}(\Sigma, I)$ be a trace language. If the syntactic ordered monoid of L is in $[\![x^\omega y x^\omega \leq x^\omega]\!] ⓜ \mathbf{V}$, then $L \in \operatorname{Pol} \mathcal{V}$.*

Proof. Let $K = \pi_I^{-1}(L)$. By Theorem 3.6 we can conclude that

$$K = \bigcup_{1 \le i \le m} K_{i,0}\ a_{i,1}\ K_{i,1} \cdots a_{i,n_i}\ K_{i,n_i}$$

for $m, n_1, \ldots, n_m \in \mathbb{N}$, $a_{i,j} \in \Sigma$ and $K_{i,j} \subseteq \Sigma^*$ such that $M(K_{i,j}) \in \mathbf{V}$. By Lemma 3.4 we have $\pi_I^{-1}\pi_I(K_{i,j}) = K_{i,j}$ and by Lemma 3.5 we can conclude that the syntactic monoid of $L_{i,j} = \pi_I(K_{i,j})$ is in $\mathbf{V}$. Hence

$$\begin{aligned} L = \pi_I(K) &= \pi_I\left(\bigcup_{1 \le i \le m} K_{i,0}\ a_{i,1}\ K_{i,1} \cdots a_{i,n_i}\ K_{i,n_i}\right) \\ &= \bigcup_{1 \le i \le m} \pi_I\left(K_{i,0}\right)\ a_{i,1}\ \pi_I\left(K_{i,1}\right) \cdots a_{i,n_i}\ \pi_I\left(K_{i,n_i}\right) \\ &= \bigcup_{1 \le i \le m} L_{i,0}\ a_{i,1}\ L_{i,1} \cdots a_{i,n_i}\ L_{i,n_i} \in \operatorname{Pol} \mathcal{V}. \quad \square \end{aligned}$$

Corollary 3.8. *Let* $\mathbf{J_1} \subseteq \mathbf{V} \subseteq \mathbf{Com}$ *be a variety of finite monoids and let* $\mathcal{V}$ *be the corresponding class of trace languages. Then* Pol $\mathcal{V}$ *corresponds to* $[\![x^\omega y x^\omega \le x^\omega]\!] ⓜ \mathbf{V}$.

Let $\mathcal{V}$ be a class of trace languages. By coPol $\mathcal{V}$ we denote the class of trace languages L whose complement $\overline{L}$ is in Pol $\mathcal{V}$. Since the syntactic ordered monoid of the complement $\overline{L}$ of a trace language L is $(M(\overline{L}), \le_{\overline{L}}) = (M(L), \le_L^{-1})$ we obtain the following corollary.

Corollary 3.9. *Let* $\mathbf{J_1} \subseteq \mathbf{V} \subseteq \mathbf{Com}$ *be a variety of finite monoids and let* $\mathcal{V}$ *be the corresponding class of trace languages. Then* coPol $\mathcal{V}$ *corresponds to* $[\![x^\omega y x^\omega \ge x^\omega]\!] ⓜ \mathbf{V}$.

The intersection of $[\![x^\omega y x^\omega \le x^\omega]\!] ⓜ \mathbf{V}$ and $[\![x^\omega y x^\omega \ge x^\omega]\!] ⓜ \mathbf{V}$ is the variety $[\![x^\omega y x^\omega = x^\omega]\!] ⓜ \mathbf{V}$; see [11]. This leads to the following corollary.

Corollary 3.10. *Let* $\mathbf{J_1} \subseteq \mathbf{V} \subseteq \mathbf{Com}$ *be a variety of finite monoids and let* $\mathcal{V}$ *be the corresponding class of trace languages. Then* Pol $\mathcal{V} \cap$ coPol $\mathcal{V}$ *corresponds to* $[\![x^\omega y x^\omega = x^\omega]\!] ⓜ \mathbf{V}$.

4. Unambiguous polynomial closure

For a position ν of $t \in \mathbb{M}(\Sigma, I)$ we define the following factors:

$$\begin{aligned} \operatorname{pre}(\nu) &= \{\, \chi \in t \mid \chi <_t \nu \,\} && \text{is the past of } \nu, \\ \operatorname{par}(\nu) &= \{\, \chi \in t \mid \nu \not<_t \chi,\ \chi \not<_t \nu,\ \nu \ne \chi \,\} && \text{is the parallel part of } \nu, \\ \operatorname{suf}(\nu) &= \{\, \chi \in t \mid \nu <_t \chi \,\} && \text{is the future of } \nu. \end{aligned}$$

We now have the following two factorizations: $t = \operatorname{pre}(\nu)\operatorname{label}(\nu)\operatorname{par}(\nu)\operatorname{suf}(\nu)$ and $t = \operatorname{pre}(\nu)\operatorname{par}(\nu)\operatorname{label}(\nu)\operatorname{suf}(\nu)$. We say that a product $L = L_1 a L_2$ of trace languages $L_1, L_2 \subseteq \mathbb{M}(\Sigma, I)$, $a \in \Sigma$ is *left unambiguous* if for all $t \in L$ there exists a unique position ν in t such that

- $\operatorname{label}(\nu) = a$ and
- $\operatorname{pre}(\nu) \in L_1$ and $\operatorname{par}(\nu)\operatorname{suf}(\nu) \in L_2$.

Right unambiguous products are defined symmetrically, i.e., L is *right unambiguous* if for all $t \in L$ there exists a unique position ν in t such that $\operatorname{label}(\nu) = a$ and $\operatorname{pre}(\nu)\operatorname{par}(\nu) \in L_1$ and $\operatorname{suf}(\nu) \in L_2$. A product $L_1 a L_2$ is *unambiguous* if it is left unambiguous *or* right unambiguous. Let $\mathcal{V}$ be a class of trace languages. Then we define UPol $\mathcal{V}$ as the closure of $\mathcal{V}$ under boolean operations and unambiguous products. Note that the unambiguous product for traces is not associative.

Theorem 4.1. *Let* $\mathbf{V}$ *be a variety of monoids such that* $\mathbf{J_1} \subseteq \mathbf{V}$ *and let* $\mathcal{V}$ *be the class of trace languages corresponding to* $\mathbf{V}$. *Then the syntactic monoid of every trace language in* UPol $\mathcal{V}$ *is in* $[\![x^\omega y x^\omega = x^\omega]\!] ⓜ \mathbf{V}$.

Proof. By **LI** we denote the semigroup variety $[\![x^\omega y x^\omega = x^\omega]\!]$. Let $L = L_1 a L_2$ be a left unambiguous product of L_1 and L_2 and let their syntactic monoids be $M_1, M_2 \in \mathbf{LI} ⓜ \mathbf{V}$. Let $\eta : \mathbb{M}(\Sigma, I) \to N$ be as in Lemma 3.1 and let $M(L)$ be the syntactic monoid of L and $\mu : \mathbb{M}(\Sigma, I) \to M(L)$ its syntactic homomorphism. We obtain the relational morphism $\tau = \eta \circ \mu^{-1} : M(L) \to N$. Since $\mathbf{LI} ⓜ \mathbf{V}$ forms a variety [8] and since $\mathbf{LI} ⓜ (\mathbf{LI} ⓜ \mathbf{V}) = \mathbf{LI} ⓜ \mathbf{V}$, see [10], it is sufficient to show that for all idempotents $e \in N$ we have $\tau^{-1}(e) \in \mathbf{LI}$. The theorem then follows by left–right symmetry and from the fact that classes of languages corresponding to varieties of monoids are closed under boolean operations [2]. The syntactic monoid of a language and the syntactic monoid of its complement are identical. Let L_1 and L_2 be two languages and let $\mu_1 : \mathbb{M}(\Sigma, I) \to M_1$ and $\mu_2 : \mathbb{M}(\Sigma, I) \to M_2$ be their syntactic homomorphisms. The syntactic monoid of $L_1 \cup L_2$ is a homomorphic image of $(\mu_1 \times \mu_2)\big(\mathbb{M}(\Sigma, I)\big) \subseteq M_1 \times M_2$.

Let $e^2 = e \in N$, let $x, y, u, v \in \mathbb{M}(\Sigma, I)$ such that $\eta(x) = e = \eta(y)$ and let $m \geq |\Sigma| + 1$ such that $\mu(x)^m$ is idempotent. We will show that $ux^m v \in L$ if and only if $ux^m y x^m v \in L$. The direction from left to right is the same as in Theorem 3.3. Suppose $ux^m y x^m v \in L$. Then there exists a left unambiguous factorization $ux^m y x^m v = z_1 a z_2$ with $z_1 \in L_1$ and $z_2 \in L_2$. Let $x = x_1 \cdots x_\ell$ and $y = y_1 \cdots y_\ell$ be factorizations into connected components such that $\mathrm{alph}(x_j) = \mathrm{alph}(y_j)$ for all $1 \leq j \leq \ell$. Suppose the connected component x_1 of x from the left x^m block matches with a factor of z_1 and the same connected component x_1 of x from the right x^m block matches with a factor of z_2. Since $\eta(x_1) = \eta(y_1)$ is idempotent, we can arbitrarily pump x_1 and y_1 at these two positions without changing membership to L. The possibility of pumping at both positions leads to two different factorizations of $ux^m y x^m y_1 x_1^m v \in L$. This contradicts the choice of L_1 and L_2 such that L is left unambiguous. The same argument holds for all connected components of x.

Together with Lemma 3.2 it follows that for every index $j \in \{1, \ldots, \ell\}$ of a connected component there exists $i \in \{1, 2\}$ such that the last occurrence of x_j in the left x^m block and the first occurrence of x_j in the right x^m block in $ux^m y x^m v$ are factors of z_i. Hence, the component y_j of y lies between these two occurrences of x_j, i.e., $x_j y_j x_j$ is a factor of z_i. Since $\eta(x_j y_j x_j) = \eta(x_j x_j)$, we can remove all connected components of y without changing membership to L. Therefore, $ux^m x^m v \in L$ and by idempotency of $\mu(x)^m$ we can conclude $ux^m v \in L$. $\square$

In the next section we will present a converse of the previous theorem for the variety $\mathbf{V} = \mathbf{J_1}$.

5. The variety DA

The variety **DA** is defined as $\mathbf{DA} = [\![(xy)^\omega x (xy)^\omega = (xy)^\omega]\!]$. It is known that $\mathbf{DA} = [\![x^\omega y x^\omega = x^\omega]\!] ⓜ \mathbf{J_1}$, see [8]. To be able to establish algebraic properties of **DA**, we will need some of *Green's relations*. Let M be a monoid and let $a, b \in M$. The equivalence relations $\mathcal{R}, \mathcal{L}, \mathcal{H} \subseteq M \times M$ are defined by

$$a \,\mathcal{R}\, b \;\Leftrightarrow\; aM = bM$$
$$a \,\mathcal{L}\, b \;\Leftrightarrow\; Ma = Mb$$
$$a \,\mathcal{H}\, b \;\Leftrightarrow\; a \,\mathcal{R}\, b \text{ and } a \,\mathcal{L}\, b.$$

By $\mathcal{G}_a$ for $\mathcal{G} \in \{\mathcal{R}, \mathcal{L}, \mathcal{H}\}$ we denote the $\mathcal{G}$-class of $a \in M$. All monoids $M \in \mathbf{DA}$ have the following properties [8]:

- M is $\mathcal{H}$-trivial, i.e. $|\mathcal{H}_a| = 1$ for all $a \in M$.
- $\forall a, b \in M : a \,\mathcal{R}\, ab \;\Leftrightarrow\; \mathcal{R}_a b = \mathcal{R}_a$.
- $\forall a, b \in M : a \,\mathcal{L}\, ba \;\Leftrightarrow\; b\mathcal{L}_a = \mathcal{L}_a$.

A factorization $t = t_- a t_+$ is a *left factorization* if $a \notin \mathrm{alph}(t_-)$ and if $t_- = sb$ implies $(a, b) \in D$, i.e., in this factorization a is the first occurrence of the letter a in t and no minimal element of t_- is independent of a. Symmetrically, we say that a factorization $t = t_- a t_+$ is a *right factorization* if $a \notin \mathrm{alph}(t_+)$ and if $t_+ = bs$ implies $(a, b) \in D$.

Definition 5.1. We define the relation $\equiv_{A,k} \subseteq \mathbb{M}(\Sigma, I)^2$ for $A \subseteq \Sigma$ and $k \in \mathbb{N}$:

- $t \equiv_{A,0} s$ if $\mathrm{alph}(t) \subsetneq A \supsetneq \mathrm{alph}(s)$ or $\mathrm{alph}(t) \subseteq A \supseteq \mathrm{alph}(s)$.
- $t \equiv_{A,k+1} s$ if $\mathrm{alph}(t) \subsetneq A \supsetneq \mathrm{alph}(s)$ or the following three conditions hold:
  . $\mathrm{alph}(t) = \mathrm{alph}(s) \subseteq A$.
  . For all $a \in \mathrm{alph}(t)$ and all left factorizations $t = t_- a t_+$ and $s = s_- a s_+$ the conditions $t_- \equiv_{A \setminus \{a\},k} s_-$ and $t_+ \equiv_{A,k} s_+$ hold.

. For all $a \in \mathrm{alph}(t)$ and all right factorizations $t = t_- a t_+$ and $s = s_- a s_+$ the conditions $t_- \equiv_{A,k} s_-$ and $t_+ \equiv_{A\setminus\{a\},k} s_+$ hold.

It is clear that for all $A \subseteq \Sigma$ and all $k \in \mathbb{N}$ the relation $\equiv_{A,k}$ is an equivalence relation of finite index. The analog of the following lemma in the case of words was shown in [14].

Lemma 5.2. *Let $\gamma \subseteq \mathbb{M}(\Sigma, I)^2$ be a congruence of finite index such that the monoid $\mathbb{M}(\Sigma, I)/\gamma$ is in* **DA**. *Then there exists $k \in \mathbb{N}$ such that $\equiv_{\Sigma,k} \subseteq \gamma$.*

Proof. We adapt the proof of a theorem in [14]. By $[t]$ we denote the equivalence class of t with respect to γ. Let $M = \mathbb{M}(\Sigma, I)/\gamma$. A factorization $t = t_0 a_1 t_1 \cdots a_n t_n$ is called *$\mathcal{R}$-decomposition* if:

- $[t_0] = 1 \in M$.
- $[t_0 a_1 t_1 \cdots a_i] \, \mathcal{R} \, [t_0 a_1 t_1 \cdots a_i t_i]$ for all $1 \le i \le n$.
- $[t_0 a_1 t_1 \cdots a_i t_i] \, \overline{\mathcal{R}} \left[t_0 a_1 t_1 \cdots a_i t_i a_{i+1}\right]$ for all $1 \le i < n$. Here $\overline{\mathcal{R}} = M^2 \setminus \mathcal{R}$.
- $|t_i|$ is minimal with the above properties for all $0 \le i \le n$ and where $|t_i|$ is always minimized before $|t_{i+1}|$.

From $M \in$ **DA** we can conclude $a_i \notin \mathrm{alph}(t_{i-1})$ for all $1 \le i \le n$. The $\mathcal{R}$-decomposition of a trace t is not unique. Let m be the maximum of the number of $\mathcal{R}$-classes and the number of $\mathcal{L}$-classes of M. By induction on $|A|$ we will prove that for all $A \subseteq \Sigma$ and all $t, s \in \mathbb{M}(\Sigma, I)$ the following implication holds: if $\mathrm{alph}(t) \subseteq A \supseteq \mathrm{alph}(s)$ and $t \equiv_{A,|A|m} s$ then $[t] = [s]$. If $|A| = 0$ then $A = \emptyset$ and $t = \varepsilon = s$ and hence $[t] = [s]$.

Suppose $|A| > 0$ and w.l.o.g. let $\emptyset \ne \mathrm{alph}(t) = \mathrm{alph}(s) \subseteq A$. Let $t = t_0 a_1 t_1 \cdots a_n t_n$ be an $\mathcal{R}$-decomposition. Then $n < m$ holds. We define $r_i = t_i a_i t_{i+1} \cdots a_n t_n$. From the minimality of $|t_i|$ it follows that $r_i = t_i a_{i+1} r_{i+1}$ is a left factorization. From $t \equiv_{A,|A|m} s$ we can conclude that there exists a factorization $s = s_0 a_1 s_1 \cdots a_n s_n$ satisfying $t_i \equiv_{A\setminus\{a_{i+1}\},|A|m-(i+1)} s_i$ for all $0 \le i < n$. Now $i + 1 \le n < m$ implies $t_i \equiv_{A\setminus\{a_{i+1}\},|A\setminus\{a_{i+1}\}|m} s_i$ and therefore by induction $[t_i] = [s_i]$ for all $0 \le i < n$. Hence $[t] \, \mathcal{R} \, [t_0 a_1 t_1 \cdots a_n] = [s_0 a_1 s_1 \cdots a_n]$ and thus $[s] \, M \subseteq [s_0 a_1 s_1 \cdots a_n] \, M = [t] \, M$. Starting with an $\mathcal{R}$-decomposition of s we can analogously conclude $[t] \, M \subseteq [s] \, M$ and therefore $[t] \, \mathcal{R} \, [s]$. A symmetric reasoning with $\mathcal{L}$-decompositions shows $[t] \, \mathcal{L} \, [s]$ and hence $[t] \, \mathcal{H} \, [s]$. Since M is $\mathcal{H}$-trivial we have $[t] = [s]$. This shows $\equiv_{\Sigma,k} \subseteq \gamma$ for $k \ge |\Sigma| \, m$. $\square$

It follows that every trace language L with $M(L) \in$ **DA** is the disjoint union of $\equiv_{A,k}$-classes. We define the class of trace languages $\mathcal{A} = \{ A^* \mid A \subseteq \Sigma \}$. Clearly, $\mathcal{A}$ is a subset of the trace languages corresponding to $\mathbf{J_1}$.

Lemma 5.3. *Let $A \subseteq \Sigma$ and $k \in \mathbb{N}$. Every equivalence class of $\equiv_{A,k}$ is in* UPol $\mathcal{A}$.

Proof. For $a \in \Sigma$, $t \in \mathbb{M}(\Sigma, I)$, $A \subseteq \Sigma$ and $k \in \mathbb{N}$, we define the following languages:

$$
\begin{aligned}
L(a) &= \overline{\bigcup_{(a,b)\in I} \Sigma^* b} \;\cap\; (\Sigma \setminus \{a\})^* \\
R(a) &= \overline{\bigcup_{(a,b)\in I} b \Sigma^*} \;\cap\; (\Sigma \setminus \{a\})^* \\
B(\Gamma) &= \{ s \in \mathbb{M}(\Sigma, I) \mid \mathrm{alph}(s) = \Gamma \} \\
&= \Gamma^* \;\cap\; \bigcap_{b\in\Gamma} L(b) \, b \, \Sigma^* \\
E(A, k, t) &= [t]_{\equiv_{A,k}}
\end{aligned}
$$

Clearly, we have $L(a)$, $R(a)$, $B(\Gamma) \in$ UPol $\mathcal{A}$. The set $L(a)$ contains all traces t such that $\mathrm{alph}(t) \subseteq \Sigma \setminus \{a\}$ and a is the unique maximal element of ta. The set $R(a)$ is symmetric. By induction on k we proof $E(A, k, t) \in$ UPol $\mathcal{A}$. The equivalence classes of $\equiv_{A,0}$ are A^* and its complement. For $k > 0$ the language $E(A, k, t)$ is $\overline{A^*}$ or

$$
\begin{aligned}
B(\mathrm{alph}(t)) \;\cap & \bigcap_{\substack{t = t_- a t_+ \text{ is} \\ \text{left factorization}}} \big(L(a) \cap E(A \setminus \{a\}, k-1, t_-)\big) \cdot a \cdot E(A, k-1, t_+) \\
\cap & \bigcap_{\substack{t = t_- a t_+ \text{ is} \\ \text{right factorization}}} E(A, k-1, t_-) \cdot a \cdot \big(R(a) \cap E(A \setminus \{a\}, k-1, t_+)\big)
\end{aligned}
$$

Note that all products are unambiguous and all intersections are finite. □

If $L \subseteq \mathbb{M}(\Sigma, I)$ is a language whose syntactic monoid $M(L)$ is in $\mathbf{J_1}$ then syntactic homomorphism $\mu : \mathbb{M}(\Sigma, I) \to M(L)$ decomposes into $\mathrm{alph} : \mathbb{M}(\Sigma, I) \to 2^{\Sigma}$ and a homomorphism $\mu' : 2^{\Sigma} \to M(L)$ such that $\mu = \mu' \circ \mathrm{alph}$. Let $A \in 2^{\Sigma}$ be a subset of the alphabet Σ with n letters. We have

$$\mathrm{alph}^{-1}(A) = \bigcup_{\{a_1,\ldots,a_n\}=A} A^* a_1 A^* \cdots a_n A^*,$$

where the union is taken over all permutations of the letters in A. If follows that the language variety corresponding to $\mathbf{J_1}$ is contained in $\mathrm{Pol}\,\mathcal{A}$. Together with Corollary 3.10 and Theorem 4.1 we can conclude:

Corollary 5.4. *The class of trace languages* $\mathrm{UPol}\,\mathcal{A} = \mathrm{Pol}\,\mathcal{A} \cap \mathrm{coPol}\,\mathcal{A}$ *corresponds to the variety* **DA**.

6. Temporal logic

In this section we introduce two characterizations of **DA** with temporal logics. In this paper, a *temporal formula* is a term of the form

$$\varphi ::= a \mid \neg\varphi \mid (\varphi_1 \vee \varphi_2) \mid (\varphi_1 \wedge \varphi_2) \mid \mathsf{XF}\varphi \mid \mathsf{YP}\varphi \mid \mathsf{M}\varphi \mid \mathsf{X}_a\varphi \mid \mathsf{Y}_a\varphi$$

where $a \in \Sigma$. The operators XF, YP, M, X_a and Y_a are called *temporal operators*. The letter X comes from the word neXt, Y stands for Yesterday, F for Future, P for Past and M for soMewhere. For a class $\mathcal{C} \subseteq \{\mathsf{XF}, \mathsf{YP}, \mathsf{M}, \mathsf{X}_a, \mathsf{Y}_a\}$ of temporal operators we denote $\mathrm{TL}[\mathcal{C}]$ as the set of temporal formulas where we only use temporal operators in $\mathcal{C}$. Next we define when a trace $t = (V, <, \mathrm{label})$ at a position $\nu \in V$ models a temporal formula. Boolean combinations are defined straightforwardly.

$$\begin{aligned}
t, \nu \models a &\iff \mathrm{label}(\nu) = a, \quad \text{for } a \in \Sigma.\\
t, \nu \models \mathsf{XF}\varphi &\iff \exists \chi \in V:\ \nu < \chi \ \text{ and } \ t, \chi \models \varphi.\\
t, \nu \models \mathsf{YP}\varphi &\iff \exists \chi \in V:\ \chi < \nu \ \text{ and } \ t, \chi \models \varphi.\\
t, \nu \models \mathsf{M}\varphi &\iff \exists \chi \in V:\ t, \chi \models \varphi.\\
t, \nu \models \mathsf{X}_a\varphi &\iff \exists \chi \in V:\ \nu < \chi \ \text{ and } \ t, \chi \models a \wedge \varphi \ \text{ and } \ \big(\forall \xi \in V:\ \nu < \xi < \chi \Rightarrow \mathrm{label}(\xi) \neq a\big).\\
t, \nu \models \mathsf{Y}_a\varphi &\iff \exists \chi \in V:\ \chi < \nu \ \text{ and } \ t, \chi \models a \wedge \varphi \ \text{ and } \ \big(\forall \xi \in V:\ \chi < \xi < \nu \Rightarrow \mathrm{label}(\xi) \neq a\big).
\end{aligned}$$

The usage of “alphabetic filters” (as in X_a and Y_a) was introduced in [3] for local temporal logic over traces.

An *outer* temporal formula is a boolean combination of formulas of the form $\mathsf{XF}\varphi$, $\mathsf{YP}\varphi$, $\mathsf{M}\varphi$, $\mathsf{X}_a\varphi$ or $\mathsf{Y}_a\varphi$ where φ is an arbitrary temporal formula. Next we define when a trace $t = (V, <, \mathrm{label})$ models an outer temporal formula. The operators XF, YP and M all have the same *outer* semantics.

$$\begin{aligned}
t \models \mathsf{XF}\varphi &\iff \exists \nu \in V:\ t, \nu \models \varphi.\\
t \models \mathsf{X}_a\varphi &\iff \exists \nu \in V:\ t, \nu \models a \wedge \varphi \ \text{ and } \ \big(\forall \xi \in V:\ \xi < \nu \Rightarrow \mathrm{label}(\xi) \neq a\big).\\
t \models \mathsf{Y}_a\varphi &\iff \exists \nu \in V:\ t, \nu \models a \wedge \varphi \ \text{ and } \ \big(\forall \xi \in V:\ \nu < \xi \Rightarrow \mathrm{label}(\xi) \neq a\big).
\end{aligned}$$

The idea is that when evaluating XF and X_a we start at a position in front of the trace and when evaluating YP and Y_a we start at a position behind the trace. The trace language generated by an outer temporal formula φ is

$$L(\varphi) = \{t \in \mathbb{M}(\Sigma, I) \mid t \models \varphi\}.$$

For a class of temporal operators $\mathcal{C}$ we say that a trace language $L \subseteq \mathbb{M}(\Sigma, I)$ is *expressible* in $\mathrm{TL}[\mathcal{C}]$ if there exists an outer temporal formula $\varphi \in \mathrm{TL}[\mathcal{C}]$ such that $L = L(\varphi)$.

Lemma 6.1. *Let φ be an outer temporal formula in* $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{M}]$. *Then the syntactic monoid of $L(\varphi)$ is in* **DA**.

Proof. Let m be the number of (nested) temporal operators in φ. Let $x, y, p, q \in \mathbb{M}(\Sigma, I)$, let $n > m\,|\Sigma|$ and let $t = p(xy)^n x (xy)^n q$ and $s = p(xy)^n (xy)^n q$.

We define a function $g : V_t \to V_s$ by mapping all positions of the prefix $p(xy)^n$ of t to the corresponding prefix of s and all positions of the suffix $(xy)^n q$ of t to the corresponding suffix of s. The positions of x in t between this prefix and this suffix are mapped to the corresponding positions of the first x of the suffix $(xy)^n q$ of s. Note that g is onto.

For $\ell < m$ we define a partial function $next_\ell : V_s \to V_s$ by mapping all positions of all xy in $(xy)^{(m-\ell)|\Sigma|}(xy)^{(m-\ell)|\Sigma|}$ in the center of

$$s = p(xy)^{n'} \cdot (xy)^{(m-\ell)|\Sigma|}(xy)^{(m-\ell)|\Sigma|} \cdot (xy)^{n'} q$$

with $n' = n - (m - \ell)\,|\Sigma|$ to the corresponding positions in the consecutive occurrence of xy in s.

We will show that for every temporal formula ζ with ℓ temporal operators, $\ell < m$, we have

$$t, \nu \models \zeta \;\Leftrightarrow\; s, g(\nu) \models \zeta \;\text{ for all } \nu \in V_t \text{ and} \tag{1}$$

$$s, \chi \models \zeta \;\Leftrightarrow\; s, next_\ell(\chi) \models \zeta \;\text{ for all } \chi \text{ in the domain of } next_\ell. \tag{2}$$

For $\ell = 0$ this is true since all positions are mapped to positions with the same label. Suppose $\zeta = \mathsf{XF}\psi$ and $t, \nu \models \zeta$. Then there exists a position ξ such that $\nu <_t \xi$ and $t, \xi \models \psi$. By induction we have $s, g(\xi) \models \psi$. If $g(\nu) <_s g(\xi)$ we are done for this direction. Otherwise ν is a position of x in the center of t. In this cases we have $g(\nu) <_s next_{\ell-1}(g(\xi))$ and by induction $s, g(\xi) \models \psi$ implies $s, next_{\ell-1}(g(\xi)) \models \psi$. Hence $s, g(\nu) \models \mathsf{XF}\psi$. The other direction of (1) follows similarly by using (2) if ν is a position of x in the center of t.

For (2) suppose $s, \chi \models \mathsf{XF}\psi$ for χ in the domain of $next_\ell$. Then there exists $\xi \in V_s$ such that $\chi <_s \xi$ and $s, \xi \models \psi$. If $next_\ell(\chi) <_s \xi$ we are done. Otherwise we can apply induction hypothesis to conclude $s, next_{\ell-1}(\xi) \models \psi$ since ξ is in the domain of $next_{\ell-1}$. Note that this is the reason for the factor $|\Sigma|$ in the above exponents in the definition of $next_\ell$. Since $next_\ell(\chi) <_s next_{\ell-1}(\xi)$ we have $s, next_\ell(\chi) \models \mathsf{XF}\psi$. The other direction of (2) is similar.

The case $\zeta = \mathsf{YP}\psi$ is symmetric and the case $\zeta = \mathsf{M}\psi$ is trivial since no constraint is required for the subsequent position. Boolean operations are straightforward by induction on the size of the formula. It follows that $t \in L(\varphi)$ if and only if $s \in L(\varphi)$ and hence the syntactic monoid of $L(\varphi)$ is in $\mathbf{DA} = [\![\, (xy)^\omega x (xy)^\omega = (xy)^\omega \,]\!]$. $\square$

Lemma 6.2. *Let φ be an outer temporal formula in* TL$[\mathsf{X}_a, \mathsf{Y}_a]$. *Then the syntactic monoid of $L(\varphi)$ is in* **DA**.

Proof. As in the proof of Lemma 6.1, let m be the number of (nested) temporal operators in φ. Let $x, y, p, q \in \mathbb{M}(\Sigma, I)$, let $n > m\,|\Sigma|$ and let $t = p(xy)^n x(xy)^n q$ and $s = p(xy)^n(xy)^n q$. The reason that φ cannot distinguish between t and s is that is not possible to reach a position in x in the center of t since $\mathrm{alph}(x) \subseteq \mathrm{alph}(xy)$. Therefore, all positions that are taken into account by a formula φ with m (nested) operators in $\{\mathsf{X}_a, \mathsf{Y}_a\}$ lie within the identical prefixes $p(xy)^{m|\Sigma|}$ and suffixes $(xy)^{m|\Sigma|}q$ of t and s. Hence the syntactic monoid of $L(\varphi)$ is in **DA**. $\square$

We will show that all trace languages L with $M(L) \in \mathbf{DA}$ can be expressed by a formula in TL$[\mathsf{XF}, \mathsf{YP}]$. By Lemma 5.2 it suffices to show that all equivalence classes of $\equiv_{A,k}$ are expressible in TL$[\mathsf{XF}, \mathsf{YP}]$.

Lemma 6.3. *Let $A \subseteq \Sigma$ and $k \in \mathbb{N}$. Every equivalence class of $\equiv_{A,k}$ is expressible in* TL$[\mathsf{XF}, \mathsf{YP}]$.

Proof. We introduce some macro formulas.

$$\begin{aligned}
B^*(\Gamma) &= \bigwedge_{b \notin \Gamma} \neg\mathsf{XF}b && \text{for } \Gamma \subseteq \Sigma\\
B(\Gamma) &= \bigwedge_{b \in \Gamma} \mathsf{XF}b \wedge B^*(\Gamma) && \text{for } \Gamma \subseteq \Sigma\\
\mathrm{LL}(a) &= \mathsf{XF}(a \wedge \neg\mathsf{YP}a) && \text{for } a \in \Sigma\\
\mathrm{LR}(a) &= \neg\mathrm{LL}(a) \wedge \neg(a \wedge \neg\mathsf{YP}a) && \text{for } a \in \Sigma\\
\mathrm{RR}(a) &= \mathsf{YP}(a \wedge \neg\mathsf{XF}a) && \text{for } a \in \Sigma\\
\mathrm{RL}(a) &= \neg\mathrm{RR}(a) \wedge \neg(a \wedge \neg\mathsf{XF}a) && \text{for } a \in \Sigma.
\end{aligned}$$

A trace t models $B^*(\Gamma)$ if and only if $\mathrm{alph}(t) \subseteq \Gamma$ and it models $B(\Gamma)$ if and only if $\mathrm{alph}(t) = \Gamma$. The first letter of the name in the formulas of the form XY(a) indicates the type of the factorization (*L*eft or *R*ight) and the second letter refers to the side within this factorization. These four formulas will be used to restrict positions to the left or to the right part of a left or a right factorization. For example, LR(a) is true at positions that are not before the first a and that are not the first position labeled by a.

For each equivalence class $[t]_{\equiv_{A,k}}$ we will show that there exists a formula $\varphi(A,k,t)$ such that $L(\varphi(A,k,t)) = [t]_{\equiv_{A,k}}$. For $k = 0$, we have $A^* = L(B^*(A))$ and $\overline{A^*}$ is expressed by the negation of this formula. For $a \in \Sigma$ we define the four transformations $\mathrm{Tr}^a_{\mathrm{XY}}$ for $\mathrm{XY} \in \{\mathrm{LL}, \mathrm{LR}, \mathrm{RL}, \mathrm{RR}\}$ on formulas in $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}]$:

$$\begin{aligned}
\mathrm{Tr}^a_{\mathrm{XY}}[\, b\,] &= b \wedge \mathrm{XY}(a) \quad \text{for } b \in \Sigma \\
\mathrm{Tr}^a_{\mathrm{XY}}[\, \neg\varphi\,] &= \neg\mathrm{Tr}^a_{\mathrm{XY}}[\, \varphi\,] \\
\mathrm{Tr}^a_{\mathrm{XY}}[\, \varphi_1 \vee \varphi_2\,] &= \mathrm{Tr}^a_{\mathrm{XY}}[\, \varphi_1\,] \vee \mathrm{Tr}^a_{\mathrm{XY}}[\, \varphi_2\,] \\
\mathrm{Tr}^a_{\mathrm{XY}}[\, \varphi_1 \wedge \varphi_2\,] &= \mathrm{Tr}^a_{\mathrm{XY}}[\, \varphi_1\,] \wedge \mathrm{Tr}^a_{\mathrm{XY}}[\, \varphi_2\,] \\
\mathrm{Tr}^a_{\mathrm{XY}}[\, \mathsf{XF}\varphi\,] &= \mathsf{XF}\big(\mathrm{Tr}^a_{\mathrm{XY}}[\, \varphi\,] \wedge \mathrm{XY}(a)\big) \\
\mathrm{Tr}^a_{\mathrm{XY}}[\, \mathsf{YP}\varphi\,] &= \mathsf{YP}\big(\mathrm{Tr}^a_{\mathrm{XY}}[\, \varphi\,] \wedge \mathrm{XY}(a)\big).
\end{aligned}$$

The transformations restrict formulas to special regions. For example $\mathrm{Tr}^a_{\mathrm{LL}}[\varphi]$ restricts φ to all positions before the first occurrence of a. For $k > 0$ the formula for $[t]_{\equiv_{A,k}}$ is $\varphi(A,k,t) = \neg B^*(A)$ or

$$\begin{aligned}
\varphi(A,k,t) = B(\mathrm{alph}(t)) \wedge \bigwedge_{\substack{t = t_- a t_+ \text{ is} \\ \text{left factorization}}} &\left\{\begin{array}{l} \mathrm{Tr}^a_{\mathrm{LL}}\big[\, \varphi(A \setminus \{a\}, k-1, t_-)\,\big] \wedge \\ \mathrm{Tr}^a_{\mathrm{LR}}\big[\, \varphi(A, k-1, t_+)\,\big] \end{array}\right\} \wedge \\
\bigwedge_{\substack{t = t_- a t_+ \text{ is} \\ \text{right factorization}}} &\left\{\begin{array}{l} \mathrm{Tr}^a_{\mathrm{RL}}\big[\, \varphi(A, k-1, t_-)\,\big] \wedge \\ \mathrm{Tr}^a_{\mathrm{RR}}\big[\, \varphi(A \setminus \{a\}, k-1, t_+)\,\big] \end{array}\right\} \quad \square
\end{aligned}$$

Lemma 6.4. *Let $A \subseteq \Sigma$ and $k \in \mathbb{N}$. Every equivalence class of $\equiv_{A,k}$ is expressible in* $\mathrm{TL}[\mathsf{X}_a, \mathsf{Y}_a]$.

Proof. The proof is analogous to that of Lemma 6.3 but the transformations of formulas will be more involved since we do not only have to ensure that we do not leave some restriction of a trace but also that we are able to reach this restriction. For this task we have to distinguish nested temporal operators between outermost and inner operators. When dealing with the outermost operators we will solve the task of reaching some part of the trace, and when we are treating inner temporal operators we will ensure that they stay within this part. The parts under examination are determined by left and right factorizations.

We introduce new macro formulas. The names of the macros are similar to those in the proof of Lemma 6.3 but we are using different modalities.

$$\begin{aligned}
B^*(\Gamma) &= \bigwedge_{b \notin \Gamma} \neg\mathsf{X}_b b && \text{for } \Gamma \subseteq \Sigma \\
B(\Gamma) &= \bigwedge_{b \in \Gamma} \mathsf{X}_b b \wedge B^*(\Gamma) && \text{for } \Gamma \subseteq \Sigma \\
\mathrm{LL}(a) &= \mathsf{X}_a \neg\mathsf{Y}_a a && \text{for } a \in \Sigma \\
\mathrm{LR}(a) &= \neg\mathrm{LL}(a) \wedge \neg(a \wedge \neg\mathsf{Y}_a a) && \text{for } a \in \Sigma \\
\mathrm{LP}(a) &= \neg\mathrm{LL}(a) \wedge \neg\mathsf{Y}_a a \wedge \neg a && \text{for } a \in \Sigma \\
\mathrm{RR}(a) &= \mathsf{Y}_a \neg\mathsf{X}_a a && \text{for } a \in \Sigma \\
\mathrm{RL}(a) &= \neg\mathrm{RR}(a) \wedge \neg(a \wedge \neg\mathsf{X}_a a) && \text{for } a \in \Sigma \\
\mathrm{RP}(a) &= \neg\mathsf{X}_a a \wedge \neg\mathrm{RR}(a) \wedge \neg a && \text{for } a \in \Sigma.
\end{aligned}$$

The letter P in the transformations of the form $\mathrm{XY}(a)$ stands for *P*arallel. For example, $\mathrm{LP}(a)$ is true at positions that are not before the first a and that are not after a position labeled by a and that are not labeled by a.

For $a \in \Sigma$ and for $\mathrm{XY} \in \{\mathrm{LL}, \mathrm{LR}, \mathrm{RL}, \mathrm{RR}\}$ we define the inner transformation $\mathrm{Inner}^a_{\mathrm{XY}}$ on formulas in $\mathrm{TL}[\mathsf{X}_a, \mathsf{Y}_a]$:

$$\begin{aligned}
\mathrm{Inner}^a_{\mathrm{XY}}[\, b\,] &= b \wedge \mathrm{XY}(a) \quad \text{for } b \in \Sigma \\
\mathrm{Inner}^a_{\mathrm{XY}}[\, \mathsf{X}_b\varphi\,] &= \mathsf{X}_b\big(\mathrm{Inner}^a_{\mathrm{XY}}[\, \varphi\,] \wedge \mathrm{XY}(a)\big) \\
\mathrm{Inner}^a_{\mathrm{XY}}[\, \mathsf{Y}_b\varphi\,] &= \mathsf{Y}_b\big(\mathrm{Inner}^a_{\mathrm{XY}}[\, \varphi\,] \wedge \mathrm{XY}(a)\big).
\end{aligned}$$

The description of the transformation of boolean combinations is omitted. Next, we will define the four transformations $\mathrm{Outer}^a_{\mathrm{LL}}$, $\mathrm{Outer}^a_{\mathrm{RR}}$, $\mathrm{Outer}^a_{\mathrm{LR}}$ and $\mathrm{Outer}^a_{\mathrm{RL}}$ of the outermost temporal operators:

$$\begin{aligned}
\mathrm{Outer}^a_{\mathrm{LL}}[\,\mathsf{X}_b\varphi\,] &= \mathsf{X}_b\big(\mathrm{Inner}^a_{\mathrm{LL}}[\,\varphi\,] \wedge \mathrm{LL}(a)\big)\\
\mathrm{Outer}^a_{\mathrm{LL}}[\,\mathsf{Y}_b\varphi\,] &= \mathsf{X}_a\mathsf{Y}_b\big(\mathrm{Inner}^a_{\mathrm{LL}}[\,\varphi\,] \wedge \mathrm{LL}(a)\big)\\
\mathrm{Outer}^a_{\mathrm{RR}}[\,\mathsf{X}_b\varphi\,] &= \mathsf{Y}_a\mathsf{X}_b\big(\mathrm{Inner}^a_{\mathrm{RR}}[\,\varphi\,] \wedge \mathrm{RR}(a)\big)\\
\mathrm{Outer}^a_{\mathrm{RR}}[\,\mathsf{Y}_b\varphi\,] &= \mathsf{Y}_b\big(\mathrm{Inner}^a_{\mathrm{RR}}[\,\varphi\,] \wedge \mathrm{RR}(a)\big)\\
\mathrm{Outer}^a_{\mathrm{LR}}[\,\mathsf{X}_b\varphi\,] &= \left\{\begin{array}{l}
\mathsf{X}_b\big(\mathrm{Inner}^a_{\mathrm{LR}}[\,\varphi\,] \wedge \mathrm{LP}(a)\big) \;\vee\\
\mathsf{X}_a\mathsf{Y}_b\mathsf{X}_b\big(\mathrm{Inner}^a_{\mathrm{LR}}[\,\varphi\,] \wedge \mathrm{LP}(a)\big) \;\vee\\
\big(\neg\mathsf{X}_b\big(\mathrm{LP}(a)\big) \wedge \neg\mathsf{X}_a\mathsf{Y}_b\mathsf{X}_b\big(\mathrm{LP}(a)\big) \;\wedge\\
\mathsf{X}_a\mathsf{X}_b\big(\mathrm{Inner}^a_{\mathrm{LR}}[\,\varphi\,] \wedge \mathrm{LR}(a)\big)\big)
\end{array}\right\}\\
\mathrm{Outer}^a_{\mathrm{LR}}[\,\mathsf{Y}_b\varphi\,] &= \mathsf{Y}_b\big(\mathrm{Inner}^a_{\mathrm{LR}}[\,\varphi\,] \wedge \mathrm{LR}(a)\big)\\
\mathrm{Outer}^a_{\mathrm{RL}}[\,\mathsf{X}_b\varphi\,] &= \mathsf{X}_b\big(\mathrm{Inner}^a_{\mathrm{RL}}[\,\varphi\,] \wedge \mathrm{RL}(a)\big)\\
\mathrm{Outer}^a_{\mathrm{RL}}[\,\mathsf{Y}_b\varphi\,] &= \left\{\begin{array}{l}
\mathsf{Y}_b\big(\mathrm{Inner}^a_{\mathrm{RL}}[\,\varphi\,] \wedge \mathrm{RP}(a)\big) \;\vee\\
\mathsf{Y}_a\mathsf{X}_b\mathsf{Y}_b\big(\mathrm{Inner}^a_{\mathrm{RL}}[\,\varphi\,] \wedge \mathrm{RP}(a)\big) \;\vee\\
\big(\neg\mathsf{Y}_b\big(\mathrm{RP}(a)\big) \wedge \neg\mathsf{Y}_a\mathsf{X}_b\mathsf{Y}_b\big(\mathrm{RP}(a)\big) \;\wedge\\
\mathsf{Y}_a\mathsf{Y}_b\big(\mathrm{Inner}^a_{\mathrm{RL}}[\,\varphi\,] \wedge \mathrm{RL}(a)\big)\big)
\end{array}\right\}.
\end{aligned}$$

For example, when transforming $\mathsf{X}_b\varphi$ by $\mathrm{Outer}^a_{\mathrm{LR}}$ we have to ensure that we reach the first position that is labeled by b and not before the first a. This is done by a case distinction. The first case is that the first b is parallel to the first a. The next case is that there is a b before the first a and the first b that is not before the first a is parallel to the first a. The third case is that there does not exist an b that is parallel to the first a. Therefore, the first b that is not before the first a is after this first a. All cases are disjoint.

Using the above transformations, we inductively define formulas $\varphi(A, k, t)$ in $\mathrm{TL}[\mathsf{X}_a, \mathsf{Y}_a]$ expressing $[t]_{\equiv_{A,k}}$. For $k = 0$ the formula is $B^*(A)$ or its negation. For $k > 0$ we have $\varphi(A, k, t) = \neg B^*(A)$ or

$$\begin{aligned}
\varphi(A, k, t) = B(\mathrm{alph}(t)) \;\wedge\; &\bigwedge_{\substack{t = t_- a t_+ \text{ is}\\ \text{left factorization}}} \left\{\begin{array}{l}\mathrm{Outer}^a_{\mathrm{LL}}\big[\,\varphi(A\setminus\{a\}, k-1, t_-)\,\big] \;\wedge\\ \mathrm{Outer}^a_{\mathrm{LR}}\big[\,\varphi(A, k-1, t_+)\,\big]\end{array}\right\} \;\wedge\\
&\bigwedge_{\substack{t = t_- a t_+ \text{ is}\\ \text{right factorization}}} \left\{\begin{array}{l}\mathrm{Outer}^a_{\mathrm{RL}}\big[\,\varphi(A, k-1, t_-)\,\big] \;\wedge\\ \mathrm{Outer}^a_{\mathrm{RR}}\big[\,\varphi(A\setminus\{a\}, k-1, t_+)\,\big]\end{array}\right\}. \qquad\square
\end{aligned}$$

In the next theorem we summarize the characterizations of trace languages whose syntactic monoid is in the variety **DA**.

Theorem 6.5. *Let $L \subseteq \mathbb{M}(\Sigma, I)$. Then the following are equivalent:*

(i) $M(L) \in \mathbf{DA}$.
(ii) $L \in \mathrm{UPol}\,\mathcal{A}$.
(iii) $L \in \mathrm{Pol}\,\mathcal{A}$ *and* $\overline{L} \in \mathrm{Pol}\,\mathcal{A}$.
(iv) L *is expressible in* $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}]$.
(v) L *is expressible in* $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{M}]$.
(vi) L *is expressible in* $\mathrm{TL}[\mathsf{X}_a, \mathsf{Y}_a]$.

Proof. The equivalence of (i), (ii) and (iii) is Corollary 5.4. The direction "(i) ⇒ (iv)" follows from Lemmas 5.2 and 6.3. Since $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}] \subseteq \mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{M}]$ we have "(iv) ⇒ (v)". The implication "(v) ⇒ (i)" is Lemma 6.1. The direction "(i) ⇒ (vi)" follows from Lemmas 5.2 and 6.4 and the implication "(vi) ⇒ (i)" is Lemma 6.2. $\square$

7. Conclusion

We have given an algebraic characterization of Pol $\mathcal{V}$ and Pol $\mathcal{V} \cap$ coPol $\mathcal{V}$ in the case that $\mathcal{V}$ corresponds to a variety of commutative monoids that contains the monoid $(2^{\Sigma}, \cup, \emptyset)$ over subsets the alphabet Σ. We have also shown that all trace languages in UPol $\mathcal{V}$ satisfy a particular algebraic property if $\mathcal{V}$ corresponds to a variety that contains the monoid $(2^{\Sigma}, \cup, \emptyset)$. Furthermore, this property is sufficient for UPol $\mathcal{V}$ if $\mathcal{V}$ corresponds to $\mathbf{J_1}$. This leads to two language-theoretic characterizations of the variety **DA**: Pol $\mathcal{A} \cap$ coPol $\mathcal{A}$ and UPol $\mathcal{A}$ where $\mathcal{A} = \{ A^* \mid A \subseteq \Sigma \}$. Then we have given two logical characterizations of **DA**: the fragments TL[XF, YP] and TL[X_a, Y_a] and we have shown that additionally allowing the operator M does not change the expressive power of the first fragment.

Two interesting open problems are whether it is possible to prove that the algebraic characterization of Pol $\mathcal{V}$ also holds for larger classes of trace languages $\mathcal{V}$ and whether it is possible to give a language-theoretic characterization of Pol $\mathcal{V} \cap$ coPol $\mathcal{V}$ in terms of disjoint unions of unambiguous polynomials as for words.

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